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# Effect of Angle-of-Attack Oscillation on the Stability of Liquid Films


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## Introduction

To derive the maximum benefit of a melting ablator, or a transpiration coolant on a re-entry body, the liquid layer should remain in contact with the re-entry body, and its removal should be in the form of vapor only. There are a number of instability mechanisms that may lead to the loss of the liquid layer before deriving the benefit of its vaporization, depending on the position of the liquid layer on the body. Away from the stagnation region, the Kelvin-Helmholtz[1] and Craik-Benjamin[2] instability mechanisms, among others, may be important. In the stagnation region, the Lamb-Taylor mechanism[3,4] is important because it is characterized by high deceleration forces acting outwards from the liquid layer. At zero angle of attack, the liquid has little relative motion, and the deceleration force has no oscillations. At angle of attack, the pressure at the windward side is higher than that at the leeward side, and hence relative motion of the liquid exists in the stagnation region due to the pressure gradient. Moreover, an oscillatory component of deceleration is produced due to angle-of-attack oscillations. The purpose of this Note is to show that this oscillatory component of deceleration is destabilizing.

Before proceeding with the analysis, let us discuss some of the pertinent work on the Lamb-Taylor instability mechanism. Lamb[3] and then Taylor[4] showed that, in the absence of surface tension and viscosity, a liquid layer in an acceleration field directed outwards from the liquid, is inherently unstable for disturbances of all wave lengths. Bellman and Pennington[5] showed that surface tension stabilizes wave numbers above the cutoff wave number $k_c = (\rho g/T)^{1/2}$, where $g$ is the acceleration normal to the liquid surface, and $\rho$ and $T$ are the liquid density and surface tension, respectively. They also showed that viscosity reduces the amplification rate, but could never by itself make it go to zero for any finite wave length.

Although the analysis of Ref. 5 shows that disturbances with wave numbers greater than $k_c$ are fully stabilized by surface tension (i.e., only oscillatory motion with time-independent amplitude is possible), Emmons, Chang and Watson[6] observed experimentally that these waves oscillated with an ever increasing amplitude. The acceleration was provided in their experiment by stretched rubber bands. They attempted to explain this instability using nonlinear analysis. However, Nayfeh[7] pointed out that their analysis is valid only for short times and cannot be used to explain this behavior that was observed beyond the validity of the expansion. Nayfeh also showed that the nonlinearity modifies the cutoff wave number to

$$\bar{k}_c = k_c\,[1 + \tfrac{3}{8}\eta_0^2 k_c^2 + \tfrac{51}{512}\eta_0^4 k_c^4]^{1/2} \tag{1}$$

with $\eta_0$ the initial amplitude. Above this wave number disturbances oscillate with time-independent amplitudes though with amplitude-dependent frequencies. Hence, he pointed out that the physical model consisting of a liquid layer in a constant acceleration field acting outwards from the liquid cannot explain the observed instability for all wave numbers above $k_c$. He suggested that the instability is due to resonances between waves in the liquid layer and the oscillatory component of acceleration. The effect of this oscillatory component of deceleration on the liquid layer stability is discussed in the next section.

## Stability Analysis

In what follows, the effect of this deceleration on the stability of a two-dimensional liquid layer of depth $h$ is investigated. The liquid is assumed to be inviscid and initially quiescent so that its subsequent motion can be represented by a potential function. Moreover, the effects of the external gas are neglected. A Cartesian $x$-$y$ coordinate system is employed with the $x$ axis in the plane of the undisturbed surface, and the $y$ axis normal to this surface. If the surface is disturbed, then the potential function $\phi(x,y,t)$ representing the motion due to this disturbance satisfies the equation

$$\nabla^2\phi = 0 \text{ for } -\infty < x < \infty \text{ and } -h \leq y \leq \eta \tag{2}$$

where $\eta(x,t)$ is the surface disturbance. The normal velocity at the body vanishes, and hence

$$\phi_y(x, -h,t) = 0 \tag{3}$$

At the free surface, the boundary conditions, assuming infinitesimal disturbances, are

$$\phi_y = \eta_t \quad \text{on} \quad y = 0 \tag{4}$$

$$\rho\, g(t)\eta - \rho\,\phi_t + T\eta_{xx} = 0 \quad \text{on} \quad y = 0 \tag{5}$$

where

$$g(t) = g_0 + g_a \cos 2\omega t \tag{6}$$

Since the system is linear, the initial conditions are taken to be

$$\eta(x,0) = \eta_0 \cos kx \tag{7}$$

$$\eta_t(x,0) = 0 \tag{8}$$

It can be shown that the solution of Eqs. (2–8) is

$$\eta(x,t) = \eta_0\, u(t) \cos kx \tag{9}$$

$$\phi(x,y,t) = \eta_0\, v(t) \cosh k(y + h) \cos kx \tag{10}$$

where

$$v = \dot{u}/k \sinh kh \tag{11}$$

$$\ddot{u} + [Tk^2/\rho - g(t)]k \tanh kh u = 0 \tag{12}$$

Letting $\tau = \omega t$ in Eq. (12) and using Eq. (6) gives

$$d^2u/d\tau^2 + [\mu - 2\gamma \cos 2\tau]u = 0 \tag{13}$$

where

$$\mu = \frac{(Tk^2/\rho - g_0)k \tanh kh}{\omega^2}, \quad \gamma = \frac{g_a k \tanh kh}{2\omega^2} \tag{14}$$

Equation (13) is the well-known Mathieu equation[8] with stable or unstable solutions depending on the parameters $\mu$ and $\gamma$. In the stable case, the solutions are either periodic or almost periodic. In the unstable case, on the other hand,

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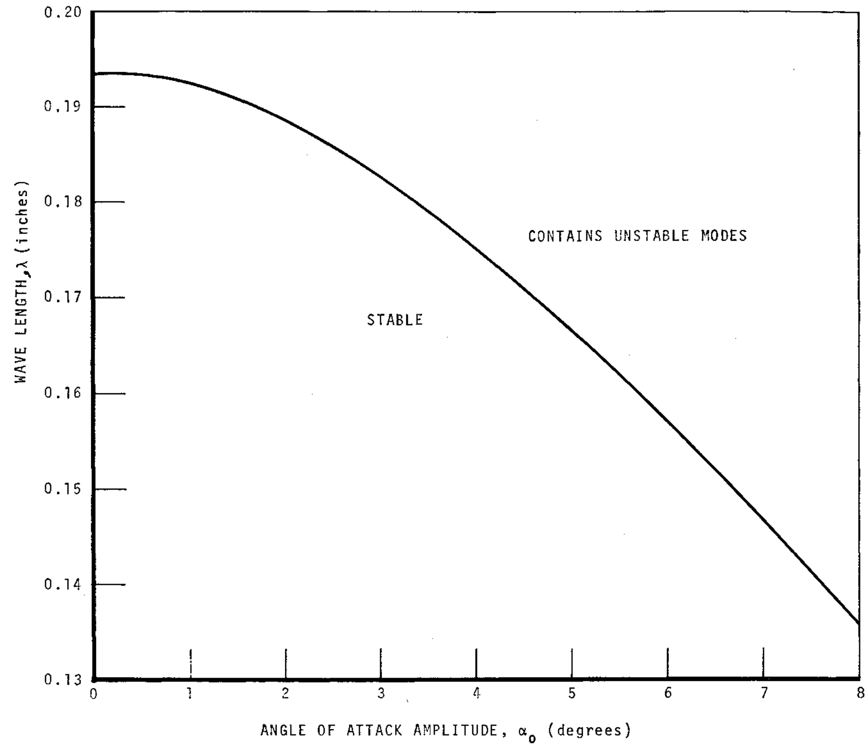


**Fig. 1 Angle-of-attack effect on the cutoff wave length for specific conditions.**

the solutions are either oscillating or nonoscillatory but with exponentially increasing amplitudes. In the next section, the parameters $\mu$ and $\gamma$ are related to the re-entry body and trajectory parameters.

### Application to Re-Entry

The deceleration of a re-entry body can be approximated by

$$g = QSC_D/m \tag{15}$$

where $Q$ is the dynamic pressure, $S$ the body cross-sectional area, $m$ the body mass, and $C_D$ its drag coefficient. If high performance sphere-cone re-entry vehicles are considered, then the drag coefficient can be approximated as the sum of a zero angle-of-attack component, and a component arising from pressure forces due to an angle of attack $\alpha$. If $\alpha < \theta$ ($\theta$ is the cone half-angle) then[9]

$$C_D \approx C_{D0} + 3\alpha^2 \tag{16}$$

where $C_{D0}$ is the drag coefficient at zero angle of attack. It can be approximated as the sum of terms arising from pressure forces, and viscous (friction) forces. At altitudes below 120 kft and for small wall to freestream temperature ratios, the following approximate expression for $C_{D0}$ can be obtained from the correlation of Ref. 9:

$$C_{D0} \approx 2 \sin^2\theta + 0.75(r_n/r_b)^2 + 1.66\, M_\infty(C/Re)^{1/2} \tag{17}$$

where $M_\infty$ is the freestream Mach number, $r_n$ and $r_b$ are the body nose and base radii, $Re$ is the Reynolds number based on body length $L$ and freestream conditions, and $C$ is the Chapman-Rubesin factor.

If damping is neglected, and the discussion is restricted to longitudinal motion,

$$\alpha = \alpha_0 \cos\omega t \tag{18}$$

where $\dot{\alpha}(0) = 0$, $\alpha_0$ is the initial angle of attack, and $\omega^2 = 2QSr_bC_{m\alpha}/I$. Here, $I$ is the transverse moment of inertia, and $C_{m\alpha}$ is the pitching moment coefficient (positive for stable vehicles). Although $Q$ and $C_{m\alpha}$ vary with altitude and hence time, their time variation is very small compared to that of the angle of attack; and hence, they can be considered constants. Using Eqs. (16) and (18) in Eq. (15) gives

$$g = g_0 + g_a \cos 2\omega t \tag{19}$$

where

$$g_0 = QS(C_{D0} + 1.5\alpha_0^2)/m, \quad g_a = 1.5\alpha_0^2 QS/m \tag{20}$$

The functions $g_0$ and $g_a$ are functions of $C_{D0}$ and $Q$, and hence are slowly varying functions of time.

Results are presented now for specific flight, body, and coolant conditions. The liquid is taken to be water at 100°C, and the liquid layer depth is 0.20 in. The body is assumed to be a sphere cone with $m$ = 95 lb, $r_n$ = 0.5 in., $r_b$ = 6.5 in., $L$ = 3.5 ft, $\theta$ = 9.3°, $I$ = 3.16 slug-ft$^2$, and $C_{m\alpha}$ = 0.034/rad. The body velocity is assumed to be 20,000 fps at 80,000 ft altitude. Most of the calculated values of $\mu$ and $\gamma$ correspond to points outside existing tabulations, and the behavior of the solutions was determined using numerical integration of Eq. (13) in conjunction with Floquet theory.[8]

Below the curve in Fig. 1 disturbances are stable, whereas above the curve, there exist modes which are unstable with large growth rates. Thus, Fig. 1 shows that the oscillatory angle of attack is destabilizing.

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# Plasma Velocity Determination by Electrostatic Probes

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## Nomenclature

$a$ = sheath radius
$a'$ = local speed of sound
$A$ = current collection surface area of probe
$c_m$ = most probable thermal speed of the particles
$e$ = basic unit of electrical charge
$I$ = current collected by a probe which has a general orientation with respect to the plasma velocity
$I_\perp$ = ion current collected by a probe perpendicular to the plasma velocity
$I_\parallel$ = ion current collected by a probe parallel to the plasma velocity
$J_n$ = Bessel function
$k$ = Boltzmann's constant
$m$ = mass of particle
$N$ = particle number density
$r$ = probe radius
$T$ = temperature
$\mathbf{v}$ = velocity of flowing plasma
$V_0$ = probe potential relative to plasma potential
$z$ = 1 + degree of ionization
$\Gamma$ = gamma function
$\tilde{\gamma}$ = incomplete gamma function
$\gamma$ = ratio of specific heats
$\gamma_0{}^2$ = $r^2/(a^2 - r^2)$
$\kappa$ = $(v/c_m)\sin\theta$
$\theta$ = orientation angle of $\mathbf{v}$ with the longitudinal axis of the probe

### *Subscripts*

$e$ = electron species
$i$ = ion species

## Introduction

IN a flowing plasma the ion current collected by a cylindrical electrostatic probe is a function of the angle between the longitudinal axis of the probe and the velocity vector of the plasma.[1–3] In the present Note this velocity dependence of ion current collection by an electrostatic probe is utilized to develop a method of determining the velocity of a flowing plasma from the Langmuir curves of two mutually perpendicular probes. This method allows one to determine the local plasma velocity simultaneously with the local measurements of the electron and ion density and the electron temperature.

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## Mathematical Development

The current collected by a moving cylindrical electrostatic probe immersed in a stationary plasma has been determined by Kanal to be given by the expression[3]

$$I = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA\, \frac{2}{\pi^{1/2}}\, e^{-\kappa^2} \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \left\{ e^{V_0} V_0{}^{-n/2} \times \Gamma\left(n + \frac{3}{2}, V_0(1 + \gamma_0{}^2)\right) J_n\, (2\kappa V_0{}^{1/2}) + \frac{a}{r}\frac{\kappa^n}{n!}\, \tilde{\gamma} \times \left(n + \frac{3}{2}, \gamma_0{}^2 V_0\right)\right\}$$

where

$$\tilde{\gamma}(\nu,\chi) = \int_0^{\chi} e^{-t}\, t^{\nu-1}\, dt \tag{2}$$

$$\Gamma(\nu,\chi) = \int_{\chi}^{\infty} e^{-t}\, t^{\nu-1}\, dt \tag{3}$$

The assumptions of a cylindrical sheath of radius $a$ and a Maxwellian velocity distribution referred to the moving coordinate system were employed. This expression may also represent the current collected by a stationary probe in a flowing plasma. If one assumes negligible sheath thickness, i.e., $a/r \to 1$, then $\gamma_0 \to \infty$, and Eq. (1) may be expressed as

$$I = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA\, \frac{2}{\pi^{1/2}}\, e^{-K^2} \sum_{n=0}^{\infty} \left[\frac{\kappa^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{4}$$

Two orientations of a cylindrical electrostatic probe relative to the flowing plasma will be considered: 1) the longitudinal axis of the probe perpendicular to the plasma velocity vector, and 2) the longitudinal axis of the probe parallel to the plasma velocity vector. The current collected by the probe in orientations 1 and 2 is denoted $I_\perp$ and $I_\parallel$, respectively. For case 1, $\kappa = v/c_m$ whereas for case 2, $\kappa = 0$. Therefore,

$$I_\perp = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA\, \frac{2}{\pi^{1/2}}\, e^{-(v/c_m)^2} \times \sum_{n=0}^{\infty} \left[\frac{(v/c_m)^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{5}$$

$$I_\parallel = \left[\frac{kT}{2\pi m}\right]^{1/2} NeA \tag{6}$$

For equal current collection surface area for the two probes, the ratio of current collected by a probe perpendicular to the plasma velocity to that collected by a probe parallel to the plasma velocity is thus given by

$$\frac{I_\perp}{I_\parallel} = \frac{2}{\pi^{1/2}}\, e^{-(v/c_m)^2} \sum_{n=0}^{\infty} \left[\frac{(v/c_m)^n}{n!}\right]^2 \Gamma\left(n + \frac{3}{2}\right) \tag{7}$$

Using the ratio test, it can be shown that the foregoing in-

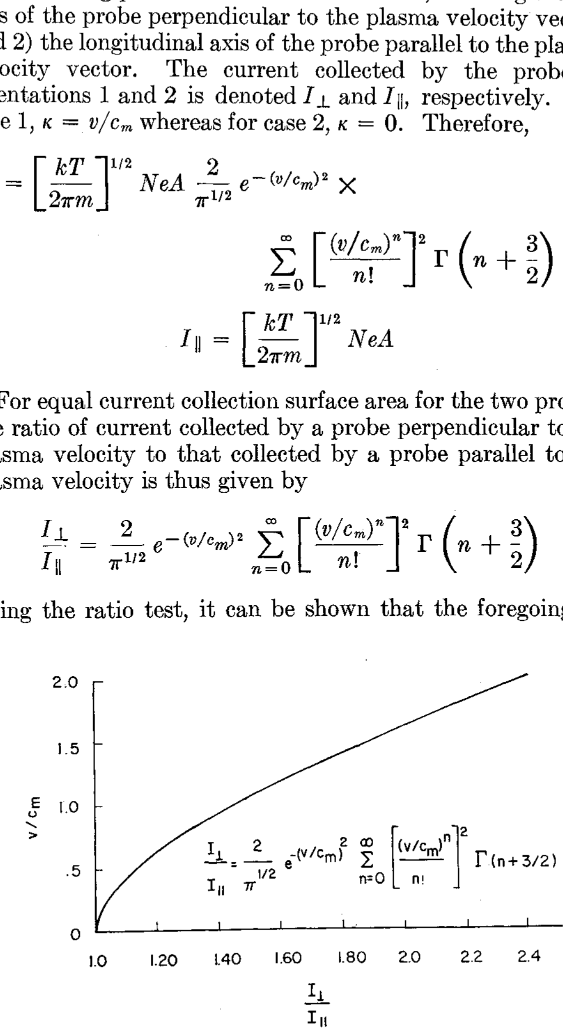


**Fig. 1 Plot of $I_\perp/I_\parallel$ vs $v/c_m$.**